Adaptive Control of Underactuated Mechanical Systems using Improved "Sigmoid Generated Fixed Point Transformation" and Scheduling Strategy

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***Abstract*—With the aim of evading the difficulties of the Lyapunov function-based techniques in the control of nonlinear systems recently the Sigmoid Generated Fixed Point Transformation (SGFPT) has been introduced. This systematic method has been presented for the generation of whole families of Fixed Point Transformations that can be used in nonlinear adaptive control of Single Input Single Output (SISO) as well as Multiple Input Multiple Output (MIMO) systems. This paper proposes a new control strategy based on the combination of the adaptive and optimal control by applying time-sharing in the SGFPT method. The scheduling strategy supports error containment by cyclic control of the different variables. Further, this paper introduces new improvements on SGFPT technique by introducing Stretched Sigmoid Functions. The efficiency of the presented control solution has been applied in the adaptive control of an underactuated mechanical system. Simulation results validate that the proposed scheme is far promising.**

***Index Terms*—Adaptive Control, Fixed Point Transformation, Robust Fixed Point Transformation (RFPT), Iterative Learning Control, Banach's Fixed Point Theorem, Sigmoid Generated Fixed Point Transformation (SGFPT), Time-sharing.**

I. Introduction

The available model of the system under control normally is complicated and can be obtained at the costs of huge efforts (e.g. [1]). In certain cases typical "ranges of operation" can be identified that gives sufficient basis for the application of the idea of "situational control" in the case of turbojet engines used in aviation [2]. However, when the characteristics of the system under control nonlinearly vary over a broad range of various conditions, the use of progressive and adaptive control approaches may be especially beneficial [3]. In a modern control solution "modeling", "control", and "diagnostic" elements are simultaneously present [4], [5]. In nonlinear systems, Lyapunov's *direct* method (or also called the *second method* of Lyapunov) have the great advantage that it does not require to analytically solve the equations of motion [6] [7]. Instead of that the uniformly continuous nature and non-positive time-derivative of a positive definite Lyapunov-function constructed of the tracking errors and the modeling errors of the systems parameters can be utilized for proving the stability of the adaptive controller. In the great majority of the applications the Lyapunov-function used to be a quadratic term defined by a positive definite symmetric matrix. In spite of its simple geometric interpretation the application of Lyapunov's second method in the practice is complicated due to the computational need of realizing this technique and also the automation of these algorithms is difficult. However many control design methodologies take their source from this theorem, for e.g., the *Model Reference Adaptive Controller (MRAC)* utilizes it for tuning the feedback parameters [8], [9], [10], [11]. Also the *Slotine-Li Adaptive Robot Controller (SLARC)* and the *Adaptive Inverse Dynamics Robot Controller (AIDRC)* control approaches are based on Lyapunov's second method for tuning the parameters of the analytical system model [12]. A review on the application of Lyapunov's second method can be found in [13]. The Robust Fixed Point Transformation (RFPT) was introduced as an alternative of Lyapunov's direct method [14], [15], [16]. The core of the theory of the Fixed Point Transformation-based approach is that it transforms the problem of the calculation of the control signal into the task of finding an appropriate fixed point of a contractive map. The fixed point is obtained by using an iterative sequence is generated by this contracting map. The RFPT-based control method assumes the existence

of the approximate model of the dynamic system that is used to calculate the control forces or torque components in case of mechanical systems, etc., based on some kinematically expressed trajectory error reduction, i.e. by the comparison of the *desired response* r^{Des} and the *actually observed response* r^{Act}. The *actually observed response* is deformed due to the exact system under control, formally $r^{Act} = f(r^{Des}, \dots)$ where the symbol "$\dots$" stand for the unknown state variables, for e.g. decoupled subsystems, disturbances, etc. Due to the model imprecisions, additive noises $r^{Act} \neq r^{Des}$. The key of the RFPT is the deformation of the input of the response function from r^{Des} to $r_\star$ using *Banach's Fixed Point Theorem* [17] in order to obtain $r^{Act} \equiv f(r_\star) = r^{Des}$.
Later the *Sigmoid Generated Fixed Point Transformation (SGFPT)* has been presented that can be used in nonlinear adaptive control of *Single Input – Single Output (SISO)* as well as *Multiple Input – Multiple Output (MIMO)* systems.

When only approximate and incomplete system models are available the other widely acknowledged approaches in nonlinear control are the so-called optimal control and the Receding Horizon Controllers. Normally the optimal controllers minimize some cost functional that is constructed of terms expressing various (often contradictory) requirements under the constraints that represent the dynamic properties of the system under control. The Receding Horizon controller is a special optimal controller that frequently redesigns the future horizon of the control so reducing the effects of modeling errors and unknown external disturbances. Optimal controllers use a cost function in order to ensure some trade off between performance and accuracy. Typical examples of application area are underactuated mechanical systems where it is impossible to drive them on an arbitrary 'trajectory' along by simultaneously precisely ensuring each state variable's position in time. In order to distribute the tracking error between the state variable's a possible solution is minimizing a cost function that is constructed as a sum of the errors. Due to a complicated mathematical framework of Lyapunov's Direct method it is far difficult to combine it with the optimal cotrollers. With this in mind in this paper we propose a strategy for replacing the optimal control by scheduling the control with time sharing in the adaptive control based on SGFPT.

II. Principles of Sigmoid Generated Fixed Point Transformation

In this section we give a short outline on the mathematical background of the *Sigmoid Generated Fixed Point Transformation*. Let us consider a monotonic increasing, bounded and smooth $g(x) : \mathbb{R} \mapsto \mathbb{R}$ *sigmoid* function. For some $K > 0$ and $D > 0$ let's consider a function $F(x) \stackrel{def}{=} g^{-1}(g(x) - K) + D$ where the inverse function of $g()$ is denoted by $g^{-1}()$. The *fixed point* of $F(x)$ is the solution of the equation $F(x) = x$. In Fig. 1 an example is shown for the case $g(x) \stackrel{def}{=} \tanh(x)$, $K = 0.5$, and $D = 0.6$.

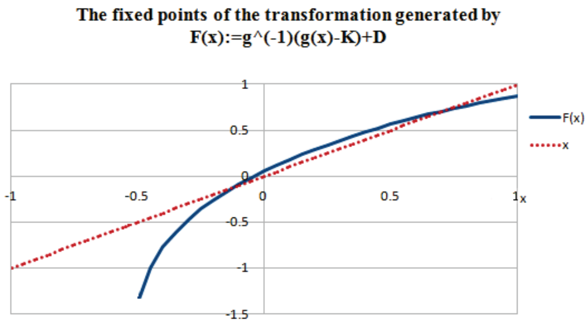

Fig. 1. The fixed points of $F(x)$

It can be observed in the figure that this function has two fixed points, at the first of them $|F'(x)| > 1$ and at the second one $|F'(x)| < 1$.

A real function $\varphi : \mathbb{R} \mapsto \mathbb{R}$ is a contraction iff there is a positive contraction constant $H < 1$ such that $|\varphi(b) - \varphi(a)| < H|b-a|$ $\forall a, b \in \mathbb{R}$. In the special case of the *differentiable functions* the contraction constant can be applied as the limit of the absolute value of the derivative as $\forall x$ $|\varphi'(x)| < H$ since according to the integral inequality

$$|\varphi(b) - \varphi(a)| = \left| \int_a^b \varphi'(x)\mathrm{d}x \right| \leq \int_a^b |\varphi'(x)|\mathrm{d}x < H|b-a| \ . \tag{1}$$

Thus, $F(x)$ is contractive around the fixed point at about $x = 0.7$. As a consequence the sequence $\{x_0, x_1 \stackrel{def}{=} F(x_0), \dots, x_{n+1} \stackrel{def}{=} F(x_n), \dots\}$ is a Cauchy sequence since $\forall L \in \mathbb{N}$

$$\begin{gathered} |x_{n+L} - x_n| = |F(x_{n-1+L}) - F(x_{n-1})| \leq \\ \leq K\,|x_{n-1+L} - x_{n-1}| \leq \dots K^n\,|x_L - x_0| \to 0 \\ \text{as } n \to \infty \ . \end{gathered} \tag{2}$$

The real numbers form complete space, thus $\exists x_* \in \mathbb{R}$ so that $x_n \to x_*$. Based on these considerations the following fixed point transformation was suggested using $F(x)$ for adaptive control if a slowly varying desired response r^{Des} is prescribed:

$$r_{n+1} = G(r_n) \stackrel{def}{=} F\left(A\left[f(r_n) - r^{Des}\right] + x_*\right) + r_n - x_* \tag{3}$$

where $A \in \mathbb{R}$ stands for a parameter, and normally, in the control applications $r_0 \stackrel{def}{=} r_0^{Des}$, that is the *desired response* in the initial control cycle. Obviously, if $r_\star$ is the solution of the control task, i.e. $f(r_\star) - r^{Des} = 0$ then $G(r_\star) = r_\star$. Since $F(x_*) = x_*$, this solution is a fixed point of the function G. In order to guarantee the convergence of the series $\{r_n\}$ function G must be contractive, i.e. the relation $\left|\frac{\mathrm{d}G}{\mathrm{d}r}\right| < 1$ must be valid. This contractivity was ensured by appropriately setting the value of the parameter A:

$$\frac{\mathrm{d}G}{\mathrm{d}r} = A\frac{\mathrm{d}F\left(A\left[f(r)-r^{Des}\right]+x_*\right)}{\mathrm{d}x}\frac{\mathrm{d}f}{\mathrm{d}r}+1 \ . \tag{4}$$

This fixed point transformation has been extended to *MIMO* systems in [18], when $f, r \in \mathbb{R}^n$, $n \in \mathbb{N}$ (n is degree of freedom of the controlled system). From this point on the iteration generates sequences as $\{r(i) \in \mathbb{R}^n | i = 0, 1, \dots\}$ by using the direction of the *response error* in the i^{th} step of the iteration as

$$h(i) \stackrel{def}{=} f(r(i)) - r^{Des}, e(i) \stackrel{def}{=} \frac{h(i)}{\|h(i)\|}, \tag{5}$$

where $\|h\|$ is the norm in the Frobenius sense:

$$\begin{aligned} r(i+1) = \tilde{G}(r(i)) \stackrel{def}{=} \\ \stackrel{def}{=} \left[F\left(A\|h(i)\| + x_*\right) - x_*\right]e(i) + r(i) \ . \end{aligned} \tag{6}$$

The solution of the control task is the fixed point of $\tilde{G}(r)$ when $f(r_\star) - r^{Des} \equiv h(i) = 0$ then $r(i+1) = r(i) = r_\star$.

Also the conditions for the stability for MIMO systems was proved in an earlier publication [18]. This solution works in bounded region of attraction around $r_\star$ that formally cannot guarantee global stabiity. By shifting the function in the horizontal and vertical direction it was assumed that for the first element of the iteration x_0 there exists x_1 for which $g(x_0) - K = g(x - D)$. This is valid for most of the x_0 values but not for each of them. In order to overcome this limitation the following stretched function is proposed

$$F(x) = B\tanh(a(x+b)) + K \tag{7}$$

with $a, b > 0$. This way allows further a more precise positioning of the function in the vicinity of the solution of the control task.

III. Adaptive Control Using Improved Sigmoid Generated Fixed Point Transformation and Scheduling Strategy

The phenomenon of swinging can be modeled by an *underactuated pendulum* according to Fig. 2: the internal degree of freedom of the human body is represented by a rotary axle $q_2\,[rad]$ with its own driving torque $Q_2\,[N\cdot m]$. The upper axle of the swing q_1 does not have any driving torque. To exemplify the applicability of the improved Sigmoid Generated Fixed Point Transformation with combination of optimal control an underactuated pendulum model serves as a benchmark problem. The dynamic model is given by Eqs. (8) and (9):

$$\begin{bmatrix} m_1L_1^2 + m_2L_1^2 + m_2L_2^2 + 2m_2L_1L_2\cos q_2 & m_2L_2^2 + m_2L_1L_2\cos q_2 \\ m_2L_2^2 + m_2L_1L_2\cos q_2 & m_2L_2^2 \end{bmatrix} \begin{bmatrix} \ddot{q}_1 \\ \ddot{q}_2 \end{bmatrix} + h = \begin{bmatrix} Q_1 \\ Q_2 \end{bmatrix} \tag{8}$$

$$h = \begin{bmatrix} -2m_2L_1L_2\sin q_2\dot{q}_1\dot{q}_2 - m_2L_1L_2\sin q_2\dot{q}_2^2 + (m_1+m_2)gL_1\sin q_1 + m_2L_2g\sin(q_1+q_2) \\ m_2L_1L_2\sin q_2\dot{q}_1^2 + m_2gL_2\sin(q_1+q_2) \end{bmatrix} \tag{9}$$

where q_1 and q_2 are rotary angles, Q_1 and Q_2 are joint torque values. In the possession of a single driving torque Q_1 *it is impossible to simultaneously precisely track nominal trajectories prescribed for both axles* as $q_1^N(t)$ and $q_2^N1(t)$. The errors of tracking the components must be distributed between q_1 and q_2. Instead of using any cost function this distribution is realized by time-sharing: for a while the motion of q_1 is controlled while q_2 is let to move "as it wants", and in the other time-slot q_2 is controlled and during this session q_1 can move without direct control. The error distribution can be manipulated by properly modifying the time-slots.

Further problem is that only an approximately known model is available for the control design (see Table I).

Parameter	Approximate value	Exact value
$m_1\,[kg]$	2.2	2
$m_2\,[kg]$	0.8	1
$L_1\,[m]$	1	1
$L_2\,[m]$	2	2
$g\,[\frac{m}{s^2}]$	10	9.81

TABLE I
THE PARAMETERS OF THE "APPROXIMATE MODEL" AND THE ACTUALLY CONTROLLED SYSTEM'S EXACT MODEL

The suggested control method at first transforms the control task into a fixed point problem then solves it via iteration: in each time-step one iteration can be done during the digital control.

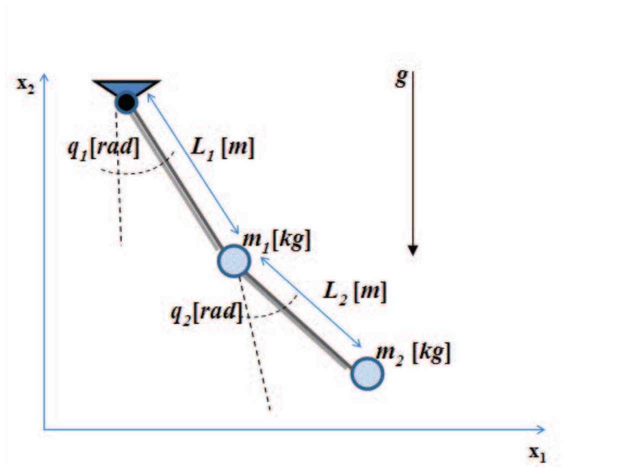

Fig. 2. Example of the *"swinging paradigm"*: in the *underactuated system* axle q_1 has no driving torque, i.e. the appropriate generalized force $Q_1\,[N] \equiv 0$. The driving torque of axle q_2 i.e. $Q_2\,[N]$ is used for the realization of a compromise in approximately tracking a nominal trajectory $q_1^N(t) \neq 0$ and $q_2^N(t) \equiv 0$. (This latter restriction is introduced for saving the body of the swinging child.)

IV. The Controller

For the control a novel fixed point transformation is used (presently under publication), the structure of the simulation is given in (Fig. 3).

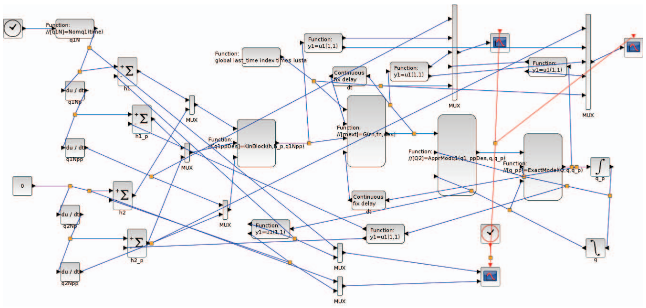

Fig. 3. The structure of the controller and the simulation

The controller's parameters are given in the "context box" of the graphical simulation. The parameter `Adaptive` must be set to 1 for adaptive, and 0 for non-adaptive control.

V. Simulation Results

For the simulation the *exact model parameters* and *approximate model parameters* were set according to I. The *kinematic trajectory tracking* was assumed to be $\ddot{q}_i^{Des} = \ddot{q}_i^{Nom} + 3\Lambda^2(q_i^{Nom} - q_i) + 3\Lambda(\dot{q}_i^{Nom} - \dot{q}_i) + \Lambda^3 \int_0^t \left(q_i^{Nom}(\tau) - q_i(\tau)\right) d\tau$ with a time constant $\Lambda = 6\,[s^{-1}]$.

The results on the trajectory tracking can be observed in Fig. 5 for the adaptive case while Fig. 4 shows the results for the non-adaptive one.

In Figs. 6 the detail of the operation of the adaptivity due to the adaptive deformation of the input can be observed.

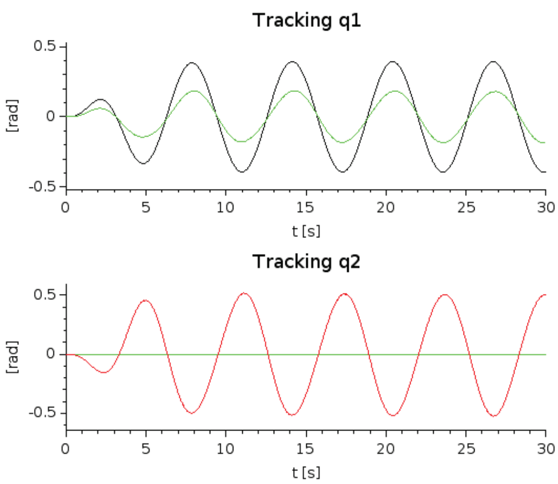

Fig. 4. Non-adaptive trajectory tracking; top: $q_1^N(t)$: black line, $q_1(t)$: green line; bottom: $q_2^N(t)$: green line, $q_2(t)$: red line

Figure 6 reveals that in the non-adaptive phase the *nominal*, the *kinematically designed "desired"* values (that contain the PID error-corrections of the nominal trajectory) and the *realized* second derivatives seriously differ to each other. (In the lack of adaptivity the "desired" and the "deformed" values are exactly identical.) In the adaptive case the "desired" and the "deformed" values considerably differ to each other, but the "realized" value has a fast convergence to the "desired" one, that is the kinematically prescribed tracking is realized in the appropriate time-slot. The same holds for the control of axle q_2 (Fig. 7).

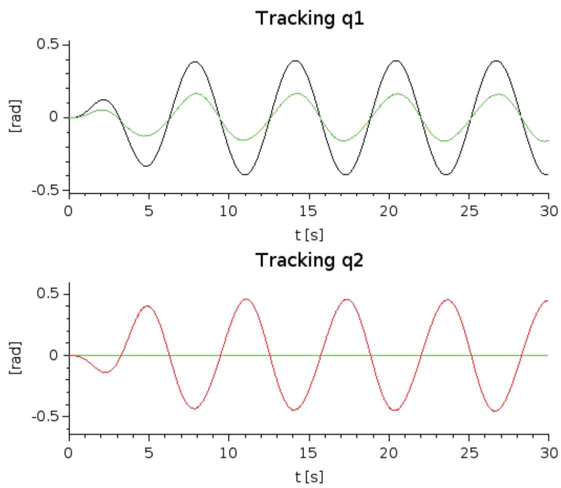

Fig. 5. Adaptive trajectory tracking; top: $q_1^N(t)$: black line, $q_1(t)$: green line; bottom: $q_2^N(t)$: green line, $q_2(t)$: red line

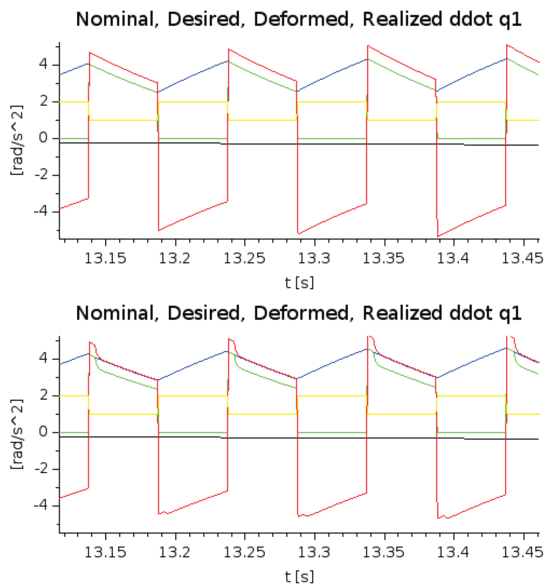

Fig. 6. Time-dependence of $\ddot{q}_1$ (zoomed-in excerpts): non-adaptive control: top, adaptive control bottom (color codes: black line: $\ddot{q}_1^N(t)$ nominal, blue line: $\ddot{q}_1^{Des}$ kinematically prescribed "desired", green line: $\ddot{q}_1^{Def}$ adaptively deformed, red line: $\ddot{q}_1$ realized (simulated), yellow line: the timer: for 1 q_1 is under control, for 2 q_2 is under control)

VI. Conclusions

In this paper a new strategy has been introduced by the combination of optimal controllers and adaptive controllers as it was suggested in [19] in the control of neuron models.

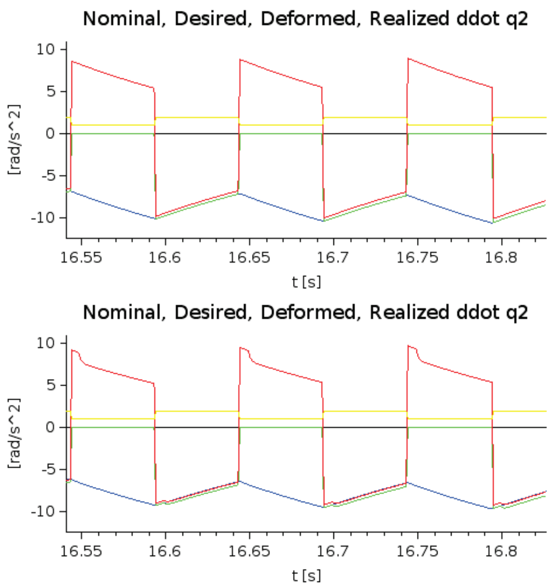

Fig. 7. Time-dependence of $\ddot{q}_2$ (zoomed-in excerpts): non-adaptive control: top, adaptive control bottom (color codes: black line: $\ddot{q}_2^N(t)$ nominal, blue line: $\ddot{q}_2^{Des}$ kinematically prescribed "desired", green line: $\ddot{q}_2^{Def}$ adaptively deformed, red line: $\ddot{q}_2$ realized (simulated), yellow line: the timer: for 1 q_1 is under control, for 2 q_2 is under control)

The proposed scheme allows to overcome the difficulties of controlling underactuated mechanical systems. For the adaptive control an improved Sigmoid Generated Fixed Transformation has been applied. The applied method is able to evade the main mathematical difficulties that are typical in the traditional approaches which originate from the use of Lyapunov's direct method. In this improved technique the sigmoid is stretched instead of simple shifting. The here suggested solution has considerable advantages in comparision with the previously used one by eliminating thats deficiencies. Optimal controllers use a cost function in order to ensure some trade off between performance and accuracy. Instead of defining a cost function, it has been realized by scheduled cyclic control of the different variables. The time-sharing is realized by using time-slots for the varibles under control. Thus the combination of the adaptive and optimal control is possible by a simple way. The performance of this new technique has been investigated in the control of an underactuated pendulum model. The performance of the proposed method has been compared with a non-adaptive PD-type control technique. Result of numerical simulations have reveald that this control strategy ensures excellent performance.

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